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LETTER TO THE EDITOR

Damage spreading and dynamic stability of kinetic Ising models

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Abstract. We investigate how the time evolution of different kinetic Ising models depends on the initial conditions of the dynamics. To this end we consider the simultaneous evolution of two identical systems subjected to the same thermal noise. We derive a master equation for the time evolution of a joint probability distribution of the two systems. This equation is then solved within an effective-field approach. By analysing the fixed points of the master equation and their stability we identify regular and chaotic phases.

The question to what extent the time evolution of a physical system depends on its initial conditions is one of the central questions in nonlinear dynamics that have lead to the discovery of chaotic behaviour [1]. In more recent years, analogous concepts have been applied to the stochastic time evolution of interacting systems with a macroscopic number of degrees of freedom. Among the simplest of such many-body systems are kinetic Ising models where the above question has been investigated by means of so-called 'damage spreading' simulations [2, 3]. In these Monte Carlo simulations two identical systems with different initial conditions are subjected to the same thermal noise, i.e. the same random numbers are used in the Monte Carlo procedure. The differences in the microscopic configurations of the two systems are then used to characterize the dynamic stability.

Later the name 'damage spreading' has also been applied to a different though related type of investigation in which the two systems are *not* identical but differ in the fact that one or several spins in one of the copies are permanently fixed in one direction. Therefore, the equilibrium properties of the two systems are different and the microscopic differences between the two copies can be related to certain thermodynamic quantities [4, 5]. Note that in this type of simulation the use of identical noise (i.e. random numbers) for the two systems is not essential but only a convenient method to reduce the statistical error. Whereas this second type of damage spreading is well understood and established as a method to numerically calculate equilibrium correlation functions, much less is known about the original problem of dynamic stability. In particular, there are no rigorous results on the transition between regular and chaotic behaviour (called the 'spreading transition'). Grassberger [6] conjectured that the spreading transition falls into the universality class of directed percolation if it does not coincide with another phase transition. This was supported by high-precision numerical simulations for the Glauber Ising model [7] where the spreading

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transition temperature is slightly lower than the equilibrium critical temperature [8]. In contrast, in the case of heat-bath dynamics the spreading temperature seems to coincide with the equilibrium critical temperature [9, 10].

In this letter we, therefore, concentrate on the original question of the stability of the stochastic dynamics in kinetic Ising models. To this end we investigate the time evolution of two *identical* systems with different initial conditions which are subjected to the same thermal noise. We derive a master equation for the joint probability distribution of the two systems and solve it within an effective-field approach. By analysing the fixed points of this equation we identify regular and chaotic phases. We find that the location of these phases in the phase diagram is sensitive to the choice of the dynamic algorithm. In particular, Glauber dynamics and heat-bath dynamics give very different dynamical phase diagrams. For the Glauber Ising model, we discuss the relation of our results to directed percolation.

We consider two identical kinetic Ising models with N sites, described by the Hamiltonian

$$H = -\frac{1}{2}\sum_{ij} J_{ij} S_i S_j - h \sum_i S_i \tag{1}$$

where S_i is an Ising variable with the values ± 1. The dynamics is given by one of the two following stochastic maps which describe Glauber dynamics

$$S_i(t+1) = \mathrm{sign}\{v[h_i(t)] - \tfrac{1}{2} + S_i(t)[\xi_i(t) - \tfrac{1}{2}]\} \tag{2a}$$

and heat-bath dynamics

$$S_i(t+1) = \mathrm{sign}\{v[h_i(t)] - \xi_i(t)\} \tag{2b}$$

with

$$v(h) = \mathrm{e}^{h/T}/(\mathrm{e}^{h/T} + \mathrm{e}^{-h/T}). \tag{3}$$

Here $h_i(t) = \sum_j J_{ij} S_j(t) + h$ is the local magnetic field at site i and (discretized) time t, $\xi_i(t) \in [0, 1)$ is a random number which is identical for both systems, and T denotes the temperature. Note that Glauber and heat-bath algorithm differ only in the way the random numbers are used to update the configuration. The transition *probabilities* v are identical for both algorithms.

In order to describe the simultaneous time evolution of two systems $H^{(1)}$ and $H^{(2)}$ with Ising spins $S_i^{(1)}$ and $S_i^{(2)}$ we define a variable $\nu_i(t)$ with the values $\nu = ++$ for $S^{(1)} = S^{(2)} = 1$, $+-$ for $S^{(1)} = -S^{(2)} = 1$, $-+$ for $-S^{(1)} = S^{(2)} = 1$, and $--$ for $S^{(1)} = S^{(2)} = -1$ which describes the state of a spin pair $(S^{(1)}, S^{(2)})$. Since we are interested in the time evolution not for a single sequence of $\xi_i(t)$ but in ξ-averaged quantities, we consider a whole ensemble of system pairs $(H^{(1)}, H^{(2)})$ and define a probability distribution

$$P(\nu_1, \ldots, \nu_N, t) = \left\langle \sum_{\nu_i(t)} \prod_i \delta_{\nu_i, \nu_i(t)} \right\rangle \tag{4}$$

where $\langle\cdot\rangle$ denotes the ensemble average. The time evolution of $P(\nu_1, \ldots, \nu_N, t)$ for a single-spin dynamical algorithm as, for example, Glauber or heat-bath dynamics is given by a master equation

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\nu_1, \ldots, \nu_N, t) = -\sum_{i=1}^{N} \sum_{\mu_i \neq \nu_i} P(\nu_1, \ldots, \nu_i, \ldots, \nu_N, t) w(\nu_i \to \mu_i)$$
$$+ \sum_{i=1}^{N} \sum_{\mu_i \neq \nu_i} P(\nu_1, \ldots, \mu_i, \ldots, \nu_N, t) w(\mu_i \to \nu_i). \tag{5}$$

Here $w(\mu_i \to \nu_i)$ is the transition probability of the spin pair $(S_i^{(1)}, S_i^{(2)})$ from state μ to ν. It is a function of the local magnetic fields $h_i^{(1)}$ and $h_i^{(2)}$ and can be calculated from (2*a*) or (2*b*) for Glauber and heat-bath dynamics, respectively.

A complete solution of the master equation (5) is, of course, out of the question. Therefore, one has to resort to approximation methods, the most obvious being mean-field approximations. A natural way to construct a mean-field theory is usually to take the range of the interaction J_{ij} to infinity at the beginning of the calculation. However, a mean-field theory constructed this way does not show the chaotic behaviour found in the Glauber Ising model at high temperatures. A more detailed analysis [11] shows that the absence of any fluctuations in the infinite-range model is responsible for this discrepancy, since the fluctuations are essential for the chaotic behaviour†.

We, therefore, develop a slightly more sophisticated effective-field approximation that retains the fluctuations, although in a quite simplistic manner. The central idea is to treat the fluctuations at different sites as statistically independent. This amounts to approximating the probability distribution $P(\nu_1, \ldots, \nu_N, t)$ by a product of identical single-site distributions P_ν,

$$P(\nu_1, \ldots, \nu_N, t) = \prod_{i=1}^{N} P_{\nu_i}(t). \tag{6}$$

Using this, the master equation (5) reduces to an equation of motion for the single-site distribution P_ν,

$$\frac{\mathrm{d}}{\mathrm{d}t} P_\nu = \sum_{\mu \neq \nu} [-P_\nu W(\nu \to \mu) + P_\mu W(\mu \to \nu)] \tag{7}$$

where

$$W(\mu \to \nu) = \langle w(\mu \to \nu) \rangle_P \tag{8}$$

is the transition probability averaged over the states ν_i of all sites according to the distribution P_ν. Note that the average magnetizations $m^{(1)}$ and $m^{(2)}$ of the two systems and the Hamming distance (also called the damage)

$$D = \frac{1}{2N} \sum_{i=1}^{N} |S_i^{(1)} - S_i^{(2)}| \tag{9}$$

which measures the distance between the two systems in phase space can be easily expressed in terms of P,

$$m^{(1)} = P_{++} + P_{+-} - P_{-+} - P_{--} \tag{10a}$$

$$m^{(2)} = P_{++} - P_{+-} + P_{-+} - P_{--} \tag{10b}$$

$$D = P_{+-} + P_{-+}. \tag{10c}$$

So far the considerations have been rather general; to be specific we will now concentrate on a two-dimensional system on a hexagonal lattice with a nearest-neighbour interaction of strength J. The external magnetic field h is set to zero. To solve the master equation (7) for the single-site distribution P we first calculate the transition probabilities $w(\mu \to \nu)$ between the states of a spin pair from one of the stochastic maps (2*a*) or (2*b*) and then average these probabilities over the states of the three neighbouring sites of a certain reference site with respect to the yet unknown distribution P. This yields the transfer rates $W(\mu \to \nu)$ which

† Damage spreading in the infinite-range *heat-bath* Ising model gives reasonable results in qualitative agreement with simulations of short-range models, see [12].

enter (7). The calculations involved are quite tedious but straightforward, and they will be presented in some detail elsewhere [11].

The resulting system of nonlinear equations for the variables P_{++}, P_{+-}, P_{-+}, and P_{--} can first be used to calculate the thermodynamics. As expected, Glauber and heat-bath dynamics give the same results. In particular, there is a ferromagnetic phase transition at a temperature T_c determined by

$$\tanh \frac{3J}{T_c} + \tanh \frac{J}{T_c} = \frac{4}{3} \tag{11}$$

which gives $T_c/J \approx 2.11$. In the ferromagnetic phase the magnetization is given by

$$m^2 = \frac{\frac{3}{4}(\tanh(3J/T) + \tanh(J/T)) - 1}{\frac{3}{4}\tanh(J/T) - \frac{1}{4}\tanh(3J/T)}. \tag{12}$$

We now discuss the time evolution of the Hamming distance D between the two systems which characterizes the stability of the dynamics. In contrast to the thermodynamics Glauber and heat-bath algorithms give very different results for the Hamming distance. We first consider the Glauber case.

The equation of motion of the Hamming distance can easily be derived from (7) and (10*c*). In the paramagnetic phase we obtain, after some algebra,

$$\frac{\mathrm{d}}{\mathrm{d}t} D = \frac{1}{2}(D - 3D^2 + 2D^3) \tanh \frac{3J}{T}. \tag{13}$$

This equation has three stationary solutions, i.e. fixed points, D^*, viz $D_1^* = 0$ which corresponds to the two systems being identical, $D_2^* = 1$ where $S^{(1)} = -S^{(2)}$ for all sites, and $D_3^* = \frac{1}{2}$ which corresponds to completely uncorrelated configurations. To investigate the stability of these fixed points we linearize (13) in $d = D - D^*$. The linearized equation has a solution $d \propto \mathrm{e}^{\lambda t}$ with $\lambda_1 = \lambda_2 = \frac{1}{2}\tanh(3J/T)$ and $\lambda_3 = -\frac{1}{4}\tanh(3J/T)$. Consequently, the only stable fixed point is $D_3^* = \frac{1}{2}$. Thus, in the paramagnetic phase the Glauber dynamics is chaotic, since two systems, starting close together in phase space (D small initially) will become separated exponentially fast with a Lyapunov exponent λ_1, eventually reaching a stationary state with an asymptotic Hamming distance $D = \frac{1}{2}$. Note, that the Lyapunov exponent λ_1 goes to zero for $T \to \infty$. Therefore, the time it takes the systems to reach the stationary state diverges with $T \to \infty$, as has also been found in simulations [13].

We now turn to the ferromagnetic phase. In order to find the fixed points of the master equation (7) we can set the magnetizations of both systems to their equilibrium values (12) from the outset. In doing so we exclude, however, all phenomena connected with the behaviour after a quench from high temperatures to temperatures below T_c. These phenomena require an investigation of the *early* time behaviour and will be analysed elsewhere [11]. For $m^{(1)} = m^{(2)} = m$ the equation of motion for the Hamming distance reads

$$\frac{\mathrm{d}}{\mathrm{d}t} D = \frac{1}{2}(D - 3D^2 + 2D^3) \tanh \frac{3J}{T} - \frac{3}{4} m^2 \left(2D \tanh \frac{J}{T} - D^2 \tanh \frac{J}{T} + D^2 \tanh \frac{3J}{T}\right). \tag{14}$$

This equation has two fixed points D^* in the interval [0, 1]. The first, $D_1^* = 0$ exists for all temperatures. The second fixed point D_3^* with $0 < D_3^* < \frac{1}{2}$ exists only for $T > T_s$ where the spreading temperature T_s is determined by

$$3m^2 \tanh \frac{J}{T_s} = \tanh \frac{3J}{T_s}. \tag{15}$$

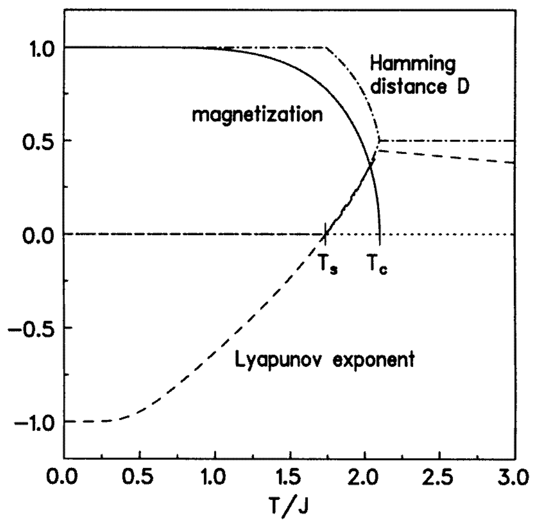

Figure 1. Magnetization m, asymptotic Hamming distance D and Lyapunov exponent λ_1 as functions of temperature for the Glauber Ising model. Below T_c the curve for D has two branches corresponding to the two systems being in the same or in different free energy valleys.

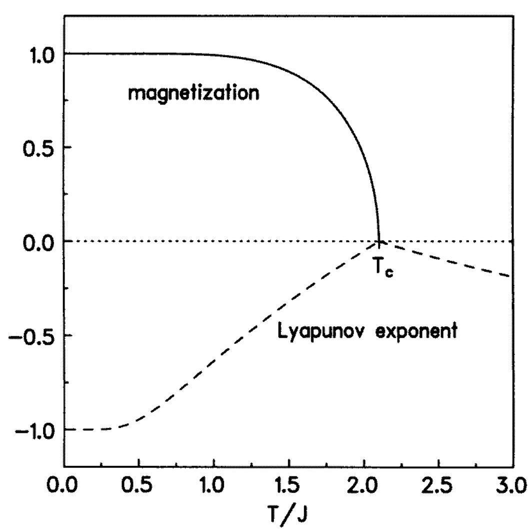

Figure 2. Magnetization m and Lyapunov exponent λ_1 as functions of temperature for the heat-bath Ising model.

This gives $T_s \approx 1.74 \approx 0.82T_c$. The stability analysis shows that $D_1^* = 0$ is stable for $T < T_s$ and unstable for $T > T_s$ with a Lyapunov exponent $\lambda_1 = \frac{1}{2}\tanh(3J/T) - \frac{3}{2}m^2\tanh(J/T)$. The fixed point D_3^* which exists only for $T > T_s$ is always stable. Consequently, we find that the Glauber dynamics is regular with the asymptotic Hamming distance being zero for temperatures smaller than the spreading temperature T_s but chaotic for $T > T_s$. Close to the spreading temperature the asymptotic Hamming distance increases linearly with $T - T_s$ which corresponds to the spreading transition being of second order with a critical exponent $\beta = 1$. In contrast to the paramagnetic phase, where the two systems become eventually completely uncorrelated, for $T_s < T < T_c$ the asymptotic Hamming distance D is always smaller than $\frac{1}{2}$ so that the two systems remain partially correlated. Directly at the spreading point the term linear in D in (14) vanishes. For small Hamming distances the equation of motion now reads $\mathrm{d}D/\mathrm{d}t \propto -D^2$ which gives a power-law behaviour $D(t) \propto t^{-\delta}$ with $\delta = 1$. Note that the values of the critical exponents, viz $\beta = \delta = 1$, are identical to the mean-field values of directed percolation.

Analogously, for $m^{(1)} = -m^{(2)} = m$ we find two fixed points, $D_2^* = 1$ which exists for all temperatures and D_4^* with $\frac{1}{2} < D_4^* < 1$ which exists for $T > T_s$ only. D_2^* is stable for temperatures $T < T_s$ and unstable for $T > T_s$ whereas D_4^* is always stable if it exists. The results for damage spreading in the Glauber Ising model within our effective-field approximation are summarized in figure 1.

We now investigate the time evolution of the damage for the Ising model with heat-bath dynamics (2*b*). After calculating the averaged transition rates $W(\mu \to \nu)$ [11] and inserting them into (7), we obtain the equation of motion for the Hamming distance D. In the paramagnetic phase it reads

$$\frac{\mathrm{d}}{\mathrm{d}t}D = \frac{3D}{4}\left[\tanh\frac{3J}{T} + \tanh\frac{J}{T} - \frac{4}{3}\right] - \frac{3D^2}{4}\left[\tanh\frac{3J}{T} + \tanh\frac{J}{T}\right] + \frac{D^3}{4}\left[\tanh\frac{3J}{T} + 3\tanh\frac{J}{T}\right]. \qquad (16)$$

This equation has only a single fixed point in the physical interval [0, 1], viz $D_1^* = 0$†. It is stable everywhere in the paramagnetic phase. Consequently, the asymptotic Hamming distance is zero for all initial conditions, and the heat-bath Ising model does not show chaotic behaviour for $T > T_c$. The Lyapunov exponent is given by

$$\lambda_1 = \frac{3}{4}\tanh\frac{3J}{T} + \frac{3}{4}\tanh\frac{J}{T} - 1 < 0.$$

The Lyapunov exponent goes to zero for $T \to T_c$, and thus at the critical temperature we again find for small Hamming distances $\mathrm{d}D/\mathrm{d}t \propto -D^2$ which gives $D(t) \propto t^{-1}$. Note that the time decay of the damage is governed *not* by the exponent $\beta/z\nu$ (here β, z, and ν are the usual equilibrium critical exponents) with the mean-field value $\frac{1}{2}$ which characterizes the decay of the magnetization. This is in agreement with numerical results [9] where the decay of the damage is found to follow a power law with a new independent exponent‡.

In the ferromagnetic phase for $m^{(1)} = m^{(2)} = m$ the equation of motion is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}D = \frac{3D}{4}\left[(1+m^2)\tanh\frac{3J}{T} + (1-3m^2)\tanh\frac{J}{T} - \frac{4}{3}\right] - \frac{3D^2}{4}\left[\tanh\frac{3J}{T} + \tanh\frac{J}{T}\right] + \frac{D^3}{4}\left[\tanh\frac{3J}{T} + 3\tanh\frac{J}{T}\right]. \qquad (17)$$

Here we also obtain only one fixed point $D_1^* = 0$ which is stable for all temperatures. The Lyapunov exponent is given by

$$\lambda_1 = \frac{3}{4}(1+m^2)\tanh\frac{3J}{T} + \frac{3}{4}(1-3m^2)\tanh\frac{J}{T} - 1 < 0.$$

† In contrast to the Glauber dynamics, the heat-bath algorithm does not preserve the symmetry with respect to a global flip of all spins. Therefore, $D = 1$ is not a fixed point here.

‡ A direct determination of the dynamical exponent z as in [9, 10] is not possible in our mean-field theory since it lacks the notion of space. However, the combination $\beta/z\nu$ can, of course, be determined from the decay of the magnetization at the critical point.

Thus, the behaviour is not chaotic and the asymptotic Hamming distance is $D = 0$. Analogously, in the ferromagnetic phase for $m^{(1)} = -m^{(2)} = m$ we obtain a single stable fixed point $D_2^* = m$. The results for damage spreading in the heat-bath Ising model within our effective-field approximation are summarized in figure 2.

In conclusion, we studied the simultaneous time evolution of two kinetic Ising models subjected to the same thermal noise by means of an effective field theory. For the heat-bath dynamics we found that two only slightly different equilibrium configurations stay close together in phase space for all times in both the paramagnetic and the ferromagnetic phase, i.e. an equilibrated heat-bath Ising model does not show chaotic behaviour. For the Glauber dynamics we found a richer behaviour. For all temperatures smaller than a spreading temperature T_s the two equilibrium configurations stay together for all times. For $T > T_s$, however, their distance increases exponentially which corresponds to chaotic behaviour. In agreement with numerical simulations for Glauber dynamics the spreading temperature T_s is not identical to the equilibrium critical temperature but slightly smaller. We determined two critical exponents of the spreading transition, viz β which characterizes the dependence of the asymptotic damage on the reduced temperature and δ which governs the time decay of the damage at the spreading point. The two exponents were found to be identical to the mean-field values of directed percolation, in agreement with Grassberger's conjecture [6].

As with any mean-field theory we have, of course, to discuss in which parameter region it correctly describes the physics of our system. Since we treated the fluctuations in a very simplistic way, viz treating fluctuations at different sites as independent, our effective-field theory will be reliable if the fluctuations are small, i.e. away from the critical point. Therefore, our theory correctly describes the high- and low-temperature behaviour, whereas it might misrepresent the details close to the critical point. It is nonetheless able to distinguish the behaviour of heat-bath and Glauber dynamics close to the spreading transition. Open questions are connected with the influence of external magnetic fields, long-range interactions, and disorder. Some investigations along these lines are in progress.

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